STABILITY OF PLANE-PARALLEL MHD FLOWS IN TRANSVERSE MAGNETIC FIELD

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We study within the framework of linear theory the stability of plane-parallel flows of a viscous, electrically conducting fluid in a transverse magnetic field. The magnetic Reynolds numbers are assumed small. The critical Reynolds number as a function of the Hartmann number is obtained over the entire range of variation of the latter. The small perturbation spectrum is studied in detail on the example of Hartmann flow. Neutral curves are constructed for symmetric and antisymmetric disturbances. The destablizing effect of a magnetic field is studied in the case of modified Couette flow. The results obtained agree with the calculations of Lock and Kakutani (where they meet) and are at variance with the results of Pavlov.

1. We examine steady flow of a viscous, incompressible, electrically conducting fluid between parallel plates in a transverse magnetic field. The magnetic Reynolds numbers are assumed small. The equation for the amplitude of the disturbance stream function $\varphi(y)$ has the form [1, 2]

$$\varphi^{IV} - 2\alpha^2\varphi'' + \alpha^4\varphi = i\alpha R\,[(u - c)(\varphi'' - \alpha^2\varphi) - u''\varphi] + G^2\varphi'' \qquad -1 \leqslant y \leqslant 1 \tag{1.1}$$

with the boundary conditions

$$\varphi(\pm 1) = \varphi'(\pm 1) = 0 \tag{1.2}$$

Here α is the wavenumber; u the velocity profile; R the Reynolds number; G the Hartmann number; $c = X + iY$ the complex perturbation phase velocity, the characteristic value of the problem. The value $Y < 0$ corresponds to decay of the perturbation; the value $Y > 0$ corresponds to growth. We take as unit length the channel halfwidth; unit velocity is the maximal stream velocity for Hartmann flow and half the relative velocity of the plates for Couette flow. The hydrodynamic stability problem reduces to analysis of the spectrum of the characteristic values of the modified Orr-Sommerfeld equation (1.1) with the boundary conditions (1.2).

Usually we restrict ourselves to studying neutral perturbations, which is sufficient for finding the critical Reynolds numbers. However for several problems, for example development of the nonlinear theory, information on the complete small-perturbation spectrum is of interest in studying the behavior of an arbitrary perturbation in the course of time. The complete small-perturbation spectrum has not been studied previously in MHD stability problems. The most complete results on stability of Hartmann flow have been obtained by Lock [1]. The stability of modified Couette flow was studied by Pavlov [3] and Kakutani [2]. They obtained different results on the dependence of the critical Reynolds number on the Hartmann number.

2. To calculate the eigenvalues of (1.1) we used a modification of the numerical method for solving the eigenvalue problem for ordinary differential equations with small parameter affiliated with the highest derivative, developed in [4]. For given G, R, α the solution of the eigenvalue problem (1.1), (1.2) yields a count-

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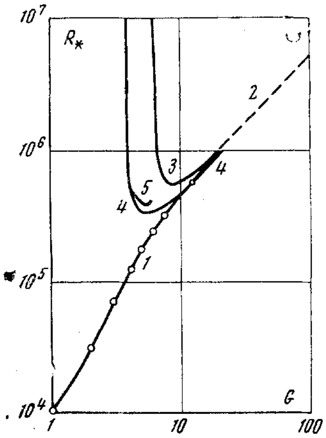

Fig. 1

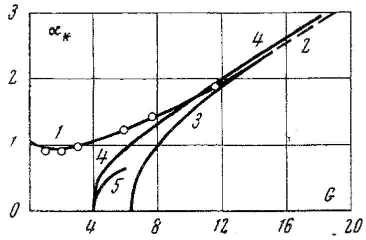

Fig. 2

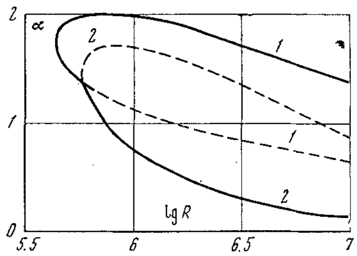

Fig. 3

able set of spectral numbers $c_n(\alpha, R, G)$. For small and large values of α we can obtain asymptotic expressions for c_n. Assuming that

$$|c| \gg \max(|u|,\ |u''|) \tag{2.1}$$

we obtain from (1.1), (1.2)

$$Y_n = -[\pi^2 (n+1)^2 + 4G^2] / 4\alpha R \tag{2.2}$$

for those small α for which (2.1) is satisfied. For Hartmann flow the values n = 1, 3, 5 ... correspond to symmetric perturbations, the values n = 2, 4, 6 ... correspond to antisymmetric perturbations. (For small even values of n, the asymptotic expression (2.2) for Y_n is approximately satisfied.) As an asymptotic expression for X_n for small α one takes the average over the cross-section

$$X_n = {}^1\!/_2 \int_{-1}^{+1} u\, dy \tag{2.3}$$

although a more exact estimate can be obtained. The spectrum numbering is made in accordance with the spectral harmonic order for small α. For large α the asymptotic expression for Y_n has the form

$$Y_n = -\alpha / R \tag{2.4}$$

just as in the case of conventional Poiseuille flow. If n >> 1, for finite R the eigenvalue spectrum in the first approximation coincides with the spectrum of a resting liquid, and therefore the instability type considered here for the velocity profile cannot be associated with large n.

Thus the region of numerical analysis is limited to the study of a finite number of spectral numbers n and a finite range of α. This circumstance makes the numerical calculations easily visualized. In studying the spectrum the construction of the relation $c_n(\alpha)$ (for fixed G and R) began with the asymptotic expressions (2.2), (2.3). Then "continuous motion" is performed up to the asymptotic values of c_n for large α (2.4). "Continuous motion" was also used to find the dependence of the critical Reynolds number R_* and the critical wavenumber α_* on G and to construct the neutral curves. The eigenvalues were found with a specified accuracy (three significant digits). The numerical calculations were made on a BÉSM-6 computer.

3. The stability of Hartmann flow

$$u = (\mathrm{ch}\, G - \mathrm{ch}\, Gy) / (\mathrm{ch}\, G - 1)$$

was studied by Lock [1] using the Heisenberg-Lin asymptotic method. He obtained the relations $R_*(G)$ and $\alpha_*(G)$ over the entire range of variation of the argument.

The numerical calculations which we made to check the numerical algorithm gave good agreement with the results of Lock. In Fig. 1 the numerical calculations are represented by the solid line (curve 1), while Lock's results are the points. The dashed curve 2 shows the asymptotic relation $R_* = 50{,}000\ G$,

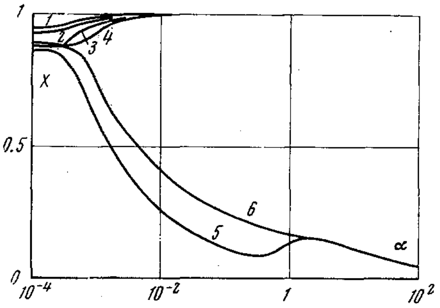

Fig. 4

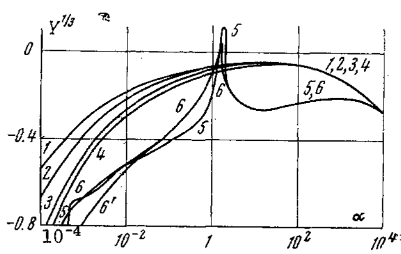

Fig. 5

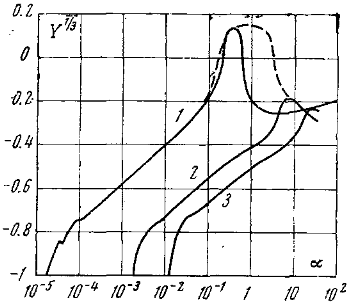

Fig. 6

constructed by Lock for the limiting case of large G, when the velocity profile degenerates to exponential form

$$u = 1 - \exp [G (y - 1)] \tag{3.1}$$

as in the case of the boundary layer with suction. Figure 2 shows the relation α_* (G). (The solid curve 1 is the present calculation, the points are Lock's data, the dashed curve 2 is Lock's asymptotic equation $\alpha_* = 0.16$ G.) Lock found that for large G even antisymmetric disturbances may become unstable. Our calculations indicate the nature of this instability. In Figs. 1 and 2 the curves 3 represent the relations R_* (G) and α_* (G) for antisymmetric disturbances. The number R_* becomes finite for $G > 6.5$; with further increase of G the minimal value $R_* = 5.4 \cdot 10^5$ is reached at G = 9.1 and it then increases again, reaching the asymptotic relation 2 for G > 15. Although R_* is smaller for symmetric disturbances, we can see from Fig. 1 that a situation may be realized in which symmetric disturbances decay while antisymmetric will be neutral or divergent.

Figure 3 shows the neutral curves for symmetric (curve 1) and antisymmetric (curve 2) disturbances for G = 10. One part of the overall neutral curve for the flow (shown by the solid curve) consists of the neutral curve for symmetric disturbances, and the other part consists of the neutral curve for antisymmetric disturbances. The overall curve has a form which is unusual for hydrodynamic stability problems. It is characteristic that along the lower branch 2 of the neutral curve the asymptotic relation $\alpha \sim 10^6/R$ is well satisfied even for $R > 10^6$. Since the critical point y_c is at a distance of order $(\alpha R)^{-1/3}$ [5] from the wall, along the lower branch of the neutral oscillations it does not approach the wall with increase of R but remains at a fixed distance from the wall, equal to 0.015 for G = 10. Along the other branches of the neutral curves the parameter αR increases and the critical point approaches the wall as $R \to \infty$.

The existence of unstable harmonics with different spectral numbers, and the arguments discussed in Section 2, suggest study of the entire small-disturbance spectrum. Let us trace the change of the small-disturbance spectrum with increase of G. In the limiting case G = 0 (Poiseuille flow) the spectrum was studied in detail in [6]. Its characteristic feature is marked separation of the spectral harmonics with increase of the wavenumber into two classes: disturbances localized near the channel wall with phase velocity approaching zero, and disturbances localized near the channel axis with phase velocity approaching unity. In the case G < 1 the spectral harmonic distribution is the same as for Poiseuille flow: Disturbances with numbers n = 1, 2, 5, 8. . . are the wall type, those with numbers n = 3, 4, 6, 7. . . are the core type. With increase of G the spectrum changes significantly.

For G = 3 the first symmetric harmonic leads to instability, just as in the Poiseuille parabolic profile case, however disturbances with spectral numbers 2 and 5, which were previously wall-type for G = 0, become core-type; the disturbance with n = 4, previously cure-type, becomes wall-type. For G = 6 the spectrum undergoes further modification and, specifically, instability is now associated with the third harmonic. Figures 4 and 5 show the relations

$$c_n (\alpha) = X_n (\alpha) + iY_n (\alpha) \text{ for } G = 10, \ R = 6 \cdot 10^5$$

For large G the velocity profile can be divided into two segments: the wall segment with dimension about 1/G, where (3.1) holds; and the core, where we can take $u \equiv 1$. Correspondingly, the spectrum of the core disturbances also corresponds with high accuracy to the spectrum of the $u \equiv 1$ profile. In this case, as apparently for any smooth convex profiles, the instability is always associated for small G with the even wall disturbances and for large G with the odd wall disturbances as well. The characteristic values at large wavenumbers for the symmetric and antisymmetric wall disturbances merge asymptotically in accordance with the localness properties [6]. Specifically, the shortwave disturbances are in practice nonzero on a small interval of α variation of order $1/\alpha$ and are independent of the nature of the homogeneous boundary conditions at the channel axis. For $\alpha_* > 2$ the characteristic values for the diverging disturbances and R_* in the symmetric and antisymmetric cases coincide, as we see in Figs. 1 and 2.

The behavior of the core disturbances also illustrates the localness properties. We see in Figs. 4 and 5 that even for small wavenumbers, when the critical point is still relatively far from the axis and the symmetry and antisymmetry conditions are not significant, the corresponding characteristic values merge by pairs. With increase of α [in the region of maxima of the relations $Y_n(\alpha)$], when the axis lies in a small vicinity of y_c, the conditions at the axis begin to have an effect and the characteristic values stratify over a small range of variation of α, and then they approach the common asymptotic relation. The stratification of the $c_n(\alpha)$ curves is not large, and therefore is not shown to avoid cluttering the figures. The localness properties for the core disturbances were analyzed in detail in [6] for the example of Poiseuille flow.

With increase of G there is an increase of the spectral number of the unstable disturbances. We see from Fig. 5 that for $G = 10$ the disturbances with $n = 5.6$ are unstable. For $G = 15$ one of the unstable disturbances corresponds to $n = 8$. For $G = 18$ one corresponds to $n = 10$. Thus, the spectral number of the unstable disturbances increases in proportion to G.

The calculations of the small-disturbance spectra for Hartmann flow were made without the term $G^2\varphi''$ in (1.1). This omission is justified provided $\alpha R \gg G^2$. The latter will not be satisfied only for $\alpha \ll \alpha_*$. In Fig. 5 the curve 6' corresponds to the sixth spectral branch, calculated with account for the term $G^2\varphi''$ in the right side of (1.1). Its influence leads to more rapid decay of the disturbances in the considered spectrum for small α. The asymptotic relations for small α for the spectral branches 6 and 6' differ significantly in accordance with (2.2).

4. Modified Couette flow

$$u = \mathrm{sh}\, Gy \,/\, \mathrm{sh}\, G$$

in a transverse magnetic field is an interesting case of destabilizing influence of a magnetic field.

The velocity profile is deformed by the magnetic field in such a way that the Reynolds stresses can lead to instability. The instability is not associated directly with the inflection point of the velocity profile [2]. Curves 4 in Figs. 1 and 2 are the relations $R_*(G)$, $\alpha_*(G)$. Curves 5 in these figures correspond to the results obtained in [2]. Our numerical calculations agree well with Kakutani's results for small values of α_*. The minimal R_* is determined more exactly than in [2]. (The author of [2] noted the inaccuracy of his values himself.) Calculation of the region of intermediate Hartmann numbers ($5 < G < 15$) now makes it possible to evaluate the stability of the subject flow over the entire range of Hartmann numbers. Kakutani concluded that the asymptotic expressions for R_*, α_* with $G > 15$ should coincide with the corresponding asymptotic expressions of Lock for Hartmann flow.

The results of the present study show that for $G > 15$ the relation $R_*(G)$ differs very little from the Lock relation. We note that the relations $R_*(G)$, $\alpha_*(G)$ are similar to the relations for antisymmetric disturbances in the case of Hartmann flow.

Figure 6 shows $Y(\alpha)$ for the most critical spectral number for $R = 10^6$ for various G. Curve 1 corresponds to $G = 5$, curve 2 is for $G = 45$, curve 3 is for $G = 100$. The corresponding calculations were made with account for the term $G^2\varphi''$ in (1.1). The breaks in the spectral branches correspond to the onset of oscillatory disturbances for small α, just as in the case of conventional Couette flow. In Fig. 6 the dashed curve shows the relation $\Pi = \max_\alpha Y(G)$ for the most critical spectral number for $R = 10^6$, which illustrates the magnetic field influence on the flow for fixed R.

Instability develops for $G = 4$, thereafter the increments of the most critical disturbances increase with increase of G, reaching a maximum at $G = 8$, and then decrease and for $G = 21$ the flow again becomes stable. In [3] the conclusion is drawn that modified Couette flow becomes unstable with respect to infinitesimal disturbances for $R \sim 25$ and $G = 1$-3. Taking into consideration the results obtained here and in

[2], we must consider that this conclusion is in error. The statement of Pavlov that the disturbances with large values of α are physically unrealistic is also invalid.

In conclusion we note that although the immediate subject of this investigation was the stability of MHD flows, the results obtained demonstrate the influence of the velocity profile shape on the small-disturbance spectrum, and the critical Reynolds numbers are actually independent of the nature of those physical effects which determine the velocity profile.

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LITERATURE CITED

1. R. C. Lock, "The stability of the flow of an electrically-conducting fluid between parallel planes under a transverse magnetic field," Proc. Roy. Soc., A 233, No. 1192 (1955).
2. T. Kakutani, "The hydromagnetic stability of the modified plane Couette flow in the presence of a transverse magnetic field," J. Phys. Soc. Japan, 19, No. 6 (1964).
3. K. B. Pavlov, "On the stability of plane Couette flow in the presence of a magnetic field," Collection: MHD and Plasma Dynamics [in Russian], Izd. AN Latv. SSR, Riga (1962).
4. M. A. Gol'dshtik and V. A. Sapozhnikov, "Stability of laminar flow in the presence of a mass-force field," Izv. AN SSSR MZhG, 3, No. 5 (1968).
5. C. C. Lin, Hydrodynamic Stability, Cambridge University Press (1966).
6. V. A. Sapozhnikov, M. A. Gol'dshtik, and V. N. Shtern, "Local properties of the velocity profile and hydrodynamic stability," Collection: 2nd Republic Conference on Aerohydrodynamics, Heat Transfer, and Mass Transfer (Summaries of Reports) [in Russian], Kiev (1969).